Que :- <u>Successive differentiation</u> :-

Let x be an independent variable and y be a dependent variable such that $y = f(x)$ then.

differentiate y w.r to x is called first order derivative of y w.r to x and is denoted

by $\frac{dy}{dx} = y_1 = y'$

Similarly if first order of y w.r to x is called second order derivative and is denoted

by $\frac{d^2y}{dx^2} = y_2 = y'''$

Similarly

The n^{th} order derivative of y is denoted

by $\frac{d^ny}{dx^n} = y_n = y^n$.

Some standard results of n^{th} order derivative :-

Theorem :- If $y = (ax+b)^m$, then

$$y_n = m(m-1) \cdots (m-n+1)a^n (ax+b)^{m+n}$$

Proof :- Let $y = (ax+b)^m$

D. w.r to x

$$\Rightarrow y_1 = m(ax+b)^{m-1}(a)$$

$$y_1 = m(ax+b)^{m-1} \cdot a$$

D. w r to x

$$\Rightarrow \quad y_2 = m(m-1)(ax+b)^{m-2} \cdot a^2$$

Continue this process

$$y_n = m(m-1)(m-2) \cdots (m-n+1) a^n (ax+b)^{m-n}$$

Cor:— If $a=1$ $b=0$ & $m=n$ then $y = x^n$

then

$$y_n = n(n-1)(n-2) - (n-n+1) \cdot (1)^n (ax+b)^{n-n}$$

$$= n(n-1)(n-2) \cdots 2(1)$$

$$= \underline{\underline{\lfloor n}}$$

Exercise 1. If $y = \hat{\sin}(\hat{\sin} x)$ Prove that $y_2 + \tan x y_1 + \cos^2 x y = 0$

Sol^n:— $y = \hat{\sin}(\sin x)$

$$\Rightarrow y_1 = \cos(\sin x) \cos x$$

$$\Rightarrow \sec x y_1 = \sqrt{1-y^2}$$

$$\Rightarrow \sec^2 x y_1^2 = 1 - y^2$$

$$\Rightarrow \sec^2 x \cdot 2y_1 y_2 - y_1^2 (2\sec x \cdot \sec x \tan x) = 0 - 2yy_1$$

$$\Rightarrow 2y_1 [\sec^2 x y_2 + \sec^2 x \tan y_1] = -2yy_1$$

$$\Rightarrow y_2 + \tan x y_1 = \frac{-1}{\sec^2 x} y$$

$$\Rightarrow \boxed{y_2 + \tan x y_1 + \cos^2 x y = 0}$$

Proved.

Que:- Show the Napier's Analogies.

Sol^n:- Napier's Analogies:

$$\tan\frac{A+B}{2} = \frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cdot\cot\frac{C}{2} \quad \text{——} (1)$$

$$\tan\frac{A-B}{2} = \frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}}\cot\frac{C}{2} \quad \text{——} (2)$$

$$\tan\frac{a+b}{2} = \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\cdot\tan\frac{c}{2} \quad \text{——} (3)$$

$$\tan\frac{a-b}{2} = \frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\tan\frac{c}{2} \quad \text{——} (4)$$

It is easy to observe that when we have got sum i.e $\tan\frac{A+B}{2}$ and $\tan\frac{a+b}{2}$ in the L.H.S then we get cosine in the R.H.S and when we have difference i.e $\tan\frac{A-B}{2}$ and $\tan\frac{a-b}{2}$ in the L.H.S the we get sine in the R.H.S equation (2) is analogous with result of plane trigonometry i.e

$$\frac{B-C}{2} = \frac{b-c}{2}\cdot\cot\frac{A}{2}.$$

Proof:- 1st Method

$$\tan\frac{A}{2} = \sqrt{\left[\frac{\sin(s-b).\sin(s-c)}{\sin s\cdot\sin(s-a)}\right]}$$

and $$\tan\frac{C}{2} = \sqrt{\left[\frac{\sin(s-a).\sin(s-b)}{\sin s.\sin(s-c)}\right]}$$

$\therefore \tan\frac{A}{2}\tan\frac{C}{2} = \frac{\sin(S-b)}{\sin S}$

Similarly $\tan\frac{B}{2}\cdot\tan\frac{C}{2} = \frac{\sin(S-a)}{\sin S}$

$\tan\frac{A}{2}\cdot\tan\frac{B}{2} = \frac{\sin(S-c)}{\sin S}$

Now $\tan\frac{A+B}{2}\cdot\tan\frac{C}{2} = \frac{\tan(A/2)+\tan(B/2)}{1-\tan(A/2)\cdot\tan(B/2)}\tan C$

$$= \frac{\frac{\sin(S-b)}{\sin S}+\frac{\sin(S-a)}{\sin S}}{1-\frac{\sin(S-c)}{\sin S}} = \frac{\sin(S-b)+\sin(S-a)}{\sin S - \sin(S-c)}$$

or $\tan\frac{A+B}{2}\cdot\tan\frac{C}{2} = \frac{2\sin\frac{2S-a-b}{2}\cos\frac{a-2}{2}}{2\sin\frac{S-S-c}{2}\cdot\cos\frac{S+S-c}{2}} = \frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}$

$[\because 2S = a+b+c]$

$\therefore \tan\frac{A+B}{2} = \frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\frac{c}{2}$ which prove (1)

Similarly we can prove the result (2)

Again we know that

$\tan\frac{a}{2} = \sqrt{\left\{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}\right\}}$

$\therefore \tan\frac{a}{2}\cdot\cot\frac{c}{2} = \sqrt{\left\{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}\right\}}$ $= \sqrt{\left\{\frac{\cos(S-A)\cos(S-B)}{-\cos S\cdot\cos(S-C)}\right\}}$ $= \frac{\cos(S-A)}{\cos(S-C)}$

Similarly $\tan\frac{a}{2}\cdot\cot\frac{b}{2} = \frac{\cos(S-B)}{\cos(S-C)}$

and $\tan\frac{a}{2}\tan\frac{b}{2} = \frac{-\cos S}{\cos(S-C)}$

Now $\tan\frac{a+b}{2}\cdot\cot\frac{c}{2} = \frac{\tan\frac{a}{2}+\tan\frac{b}{2}}{1-\tan\frac{a}{2}\tan\frac{b}{2}}$

$$\frac{\frac{\cos(S-A)}{\cos(S-C)}+\frac{\cos(S-B)}{\cos(S-C)}}{1+\frac{\cos S}{\cos(S-C)}} = \frac{\cos(S-A)+\cos(S-B)}{\cos(S-C)+\cos S}$$

or $\tan\frac{a+b}{2}\cos\frac{c}{2} = \frac{2\cos\frac{2S-A-B}{2}\cos\frac{A-B}{2}}{2\cos\frac{S-C-S}{2}+\cos\frac{S-C+S}{2}} = \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}$

$[\because A+B+C = 2S]$

$\therefore \tan\frac{a+b}{2} = \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\cdot\tan\frac{c}{2}$

Similarly we can prove that

$\tan\frac{a-b}{2} = \sin\frac{A-B}{2}\ \ \tan C$